
The Lie-Poisson Structure of the Symmetry Reduced Regularised n -Body Problem

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Abstract

This paper investigates the symmetry reduction of the regularised n -body problem. The three body problem, regularised through quaternions, is examined in detail. We show that for a suitably chosen symmetry group action the space of quadratic invariants is closed and the Hamiltonian can be written in terms of the quadratic invariants. The corresponding Lie-Poisson structure is isomorphic to the Lie algebra $\mathfrak{u}(3,3)$. Finally, we generalise this result to the n -body problem for $n > 3$.

1 Introduction

The Galilean symmetry of the n -body problem leads to the classical 9 integrals of linear momentum, centre of mass, and angular momentum. Symplectic reduction of this symmetry gives a reduced system with $3n - 5$ degrees of freedom, see, e.g. [10]. An alternative approach to reduction that avoids problems with singular reduction uses invariants of the symmetry group action. Singular reduction does occur in the n -body problem because the orbit of the symmetry group drops in dimension for collinear configurations. Using quadratic invariants leads to a Lie-Poisson structure isomorphic to $\mathfrak{sp}(2n - 2)$, as was shown using different bases of invariants in [12] and [1].

One motivation for this approach is the possibility to derive structure preserving geometric integrators for the symmetry reduced 3-body problem, as done in [1]. However, numerical integration of many body problems needs to be able to deal with binary near-collisions. The classical regularisation by squaring in the complex plane found by Levi-Civita [8] has a beautiful spatial analogue that can be formulated using quaternions [5], also see [13]. This regularisation has been used by Heggie to simultaneously regularise binary collision in the n -body problem [2]. Recently the symmetry reduction of the regularised 3-body problem has been revisited in [11], extending the classical work of Lemaitre [6]. In the present work we perform the symmetry

reduction using quadratic invariants, thus repeating [1] for the regularised problem. See [7] for some background on singular reduction. Our main result is that the symmetry reduced regularised 3-body problem has the Lie-Poisson structure of the Lie-algebra $\mathfrak{u}(3,3)$.

The paper is organised as follows. In the next section we introduce our notation of quaternions and Heggge's regularised Hamiltonian. We then treat the cases $n = 2$ (Kepler), $n = 3$ and $n \geq 4$ in turns. For the Kepler problem we show how to extend the $SO(3)$ group action on \mathbb{R}^3 to an action of a subgroup of $SO(4)$ on quaternions. Treating 3 particles amounts to redoing this construction for 3 difference vectors, and we show that for a suitable chosen group action the space of quadratic invariants is closed and the Hamiltonian can be written in terms of the quadratic invariants. The corresponding Lie-Poisson structure is $\mathfrak{u}(3,3)$. In the final section we briefly comment on how this reduction is done for an arbitrary number of particles.

2 Simultaneous regularisation of binary collisions

Let the positions of the n particles be denoted by $\mathbf{q}_i \in \mathbb{R}^3$, and the conjugate momenta by $\mathbf{p}_i \in \mathbb{R}^3$, $i = 1, \dots, n$. The translational symmetry is reduced by forming difference vectors $\mathbf{q}_{ij} = \mathbf{q}_i - \mathbf{q}_j$ and $\mathbf{p}_{ij} = \mathbf{p}_i - \mathbf{p}_j$. We follow [14] in using quaternions for the regularisation. The analogue of Levi-Civita's squaring map can then be written as

$$\mathbf{q} = \mathbf{Q} * \mathbf{Q}^*, \quad (2.1)$$

where $\mathbf{Q} = Q_0 + iQ_1 + jQ_2 + kQ_3$ and the superscript $*$ flips the sign of the k -component, $\mathbf{Q}^* = Q_0 + iQ_1 + jQ_2 - kQ_3$, see [14]. By construction the quaternion $\mathbf{Q} * \mathbf{Q}^*$ has vanishing k -component and can thus be identified with the 3-dimensional vector \mathbf{q} .

The mapping from 4-dimensional momenta \mathbf{P} to 3-dimensional momenta \mathbf{p} is given by

$$\mathbf{p} = \frac{1}{2\|\mathbf{Q}\|^2} \mathbf{Q} * \mathbf{P}^* = \frac{1}{2} \mathbf{P}^* * \bar{\mathbf{Q}}^{-1} \quad (2.2)$$

where the overbar denotes quaternionic conjugation, i.e. flipping the sign of the i , j , and k component. Note that in general the k -component of the right hand side is non-zero. One could think of the map to $\mathbf{p} \in \mathbb{R}^3$ to be a projection onto the first three components. However, it turns out to be better to impose that the last component vanishes. This condition can be written as

$$\mathbf{Q}^T K \mathbf{P} = 0 \quad \text{where} \quad K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.3)$$

Here and in the following \mathbf{Q} and \mathbf{P} are interpreted as ordinary 4-dimensional vectors; multiplication of quaternions by contrast is denoted by $*$. Equation (2.3) is the famous bi-linear relation [5]. Together (2.1) and (2.2) define a projection π from $(\mathbf{Q}, \mathbf{P}) \in T^*\mathbb{R}^4$ to $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^3$. Only when restricting to the subspace defined by the bi-linear relation (2.3) does the map π respect the symplectic structures so that

$$\{f, g\}_3 \circ \pi = \{f \circ \pi, g \circ \pi\}_4.$$

Here the two Poisson brackets $\{, \}_3$ and $\{, \}_4$ are corresponding to the two standard symplectic structures $d\mathbf{q} \wedge d\mathbf{p}$ and $d\mathbf{Q} \wedge d\mathbf{P}$, respectively, see, e.g. [4, 3].

Using the transformation to \mathbf{Q}, \mathbf{P} the Hamiltonian of the n -body problem written in terms of difference vectors and scaling time gives the regularised Hamiltonian [2]

$$\begin{aligned} H = & \frac{1}{8} \left(\frac{R_{12}R_{31}}{\mu_{23}} \mathbf{P}_{23}^T \mathbf{P}_{23} + \frac{R_{12}R_{23}}{\mu_{31}} \mathbf{P}_{31}^T \mathbf{P}_{31} + \frac{R_{23}R_{31}}{\mu_{12}} \mathbf{P}_{12}^T \mathbf{P}_{12} \right) \\ & - \frac{1}{4} \left(\frac{R_{23}}{m_1} (\mathbf{Q}_{31} * \mathbf{P}_{31}^*)^T (\mathbf{Q}_{12} * \mathbf{P}_{12}^*) + \frac{R_{31}}{m_2} (\mathbf{Q}_{12} * \mathbf{P}_{12}^*)^T (\mathbf{Q}_{23} * \mathbf{P}_{23}^*) + \frac{R_{12}}{m_3} (\mathbf{Q}_{23} * \mathbf{P}_{23}^*)^T (\mathbf{Q}_{31} * \mathbf{P}_{31}^*) \right) \\ & - (m_2 m_3 R_{31} R_{12} + m_3 m_1 R_{12} R_{23} + m_1 m_2 R_{23} R_{31}) - h R_{23} R_{31} R_{12}. \end{aligned} \quad (2.4)$$

where $R_{ij} = \mathbf{Q}_{ij}^T \mathbf{Q}_{ij} = \|\mathbf{q}_{ij}\|$ and $\mu_{ij} = \frac{m_i m_j}{m_i + m_j}$ is the reduced mass of particles i and j .

3 The Kepler Problem $n = 2$

As mentioned in the introduction this case has been treated extensively in the literature [5, 3, 4], but we briefly treat it first to establish our notation and important Lemmas needed for the case with 3 or more masses. For $n = 2$ there is only a single difference vector $\mathbf{q}_{12} = \mathbf{q}_1 - \mathbf{q}_2$, similarly for \mathbf{p} . For ease of notation, in this section we are writing \mathbf{q} for \mathbf{q}_{12} , similarly for \mathbf{p} , and the corresponding quaternions \mathbf{Q} and \mathbf{P} .

For $n = 2$ the time scaling reduces the problem to the harmonic oscillator since the more complicated cross-terms in the kinetic energy vanish, so that

$$H = \frac{1}{8\mu} |\mathbf{P}|^2 - m_1 m_2 - h |\mathbf{Q}|^2.$$

The $SO(3)$ symmetry acting on pairs of difference vectors in $\mathbb{R}^3 \times \mathbb{R}^3$ is the diagonal action $\Phi_R : (\mathbf{q}, \mathbf{p}) \mapsto (R\mathbf{q}, R\mathbf{p})$ for $R \in SO(3)$. This is a symplectic map whose momentum map is the cross product $\mathbf{q} \times \mathbf{p}$. Which linear symplectic action Ψ_S of (a subgroup of) $SO(4)$ acting on $\mathbb{R}^4 \times \mathbb{R}^4$ projects to Φ_R under π ?

Lemma 3.1. *The diagonal action $\Psi_S : (\mathbf{Q}, \mathbf{P}) \mapsto (S\mathbf{Q}, S\mathbf{P})$ for $S \in G$ a subgroup of $SO(4)$ with $G \cong SU(2) \times SO(2)$ projects to the action of Φ_R under π . In other words, the diagram*

$$\begin{array}{ccc} T^*\mathbb{R}^3 & \xleftarrow{\pi} & T^*\mathbb{R}^4 \\ \Phi_R \downarrow & & \Psi_S \downarrow \\ T^*\mathbb{R}^3 & \xleftarrow{\pi} & T^*\mathbb{R}^4 \end{array}$$

commutes.

Proof. Let the rotation $R \in SO(3)$ be given by $R = \exp At$ for some $A \in \text{Skew}(3)$. We assume that we can write $S = \exp Bt$ for $B \in \text{Skew}(4)$. The diagram states that $\Phi_R \circ \pi = \pi \circ \Psi_S$. Linearising at the identity, i.e. differentiating with respect to t and setting $t = 0$, and using that Φ_S leaves the norm of quaternions unchanged gives

$$A(\mathbf{Q} * \mathbf{P}^*) = (B\mathbf{Q}) * \mathbf{P}^* + \mathbf{Q} * (B\mathbf{P})^*$$

from the momenta (2.2), and the same equation with \mathbf{P} replaced by \mathbf{Q} from the positions (2.1). For given $A = \hat{\mathbf{L}}$ with arbitrary $\mathbf{L} = (L_x, L_y, L_z)^t$ and the usual hat-map from \mathbb{R}^3 to $\text{Skew}(3)$, the general solution can be written as $B = \frac{1}{2}(\text{Isoc}(\hat{\mathbf{L}}) + \tau K)$, where $\text{Isoc}(\hat{\mathbf{L}}) = \begin{pmatrix} \hat{\mathbf{L}} & -\mathbf{L} \\ \mathbf{L}^t & 0 \end{pmatrix}$ and parameter τ . The subgroup G contains the subgroup of isoclinic rotations $\exp(\text{Isoc}(A)) = \cos \omega I_4 + \omega^{-1} \sin \omega \text{Isoc}(A)$ where $\omega^2 = \frac{1}{2} \text{Tr} AA^t$. They form a subgroup since the corresponding generators $\text{Isoc}(A)$ form an algebra with $[\text{Isoc}(\hat{\mathbf{a}}), \text{Isoc}(\hat{\mathbf{b}})] = 2 \text{Isoc}([\hat{\mathbf{a}}, \hat{\mathbf{b}}]) = 2 \text{Isoc}(\widehat{\mathbf{a} \times \mathbf{b}})$ for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The corresponding group of left-isoclinic rotation matrices $\exp(\text{Isoc}(\hat{\mathbf{a}}))$ has a composition law given by left-multiplication of unit quaternions with imaginary part proportional to \mathbf{a} . The whole group G is obtained by multiplying the general left-isoclinic rotation $\exp(\text{Isoc}(A))$ with the special right-isoclinic rotation $\exp(K\tau)$. These two commute, since $\text{Isoc}(A)$ and K commute. The group $\exp(K\tau)$ is isomorphic to $SO(2)$, so G is isomorphic of $SO(3) \times SO(2)$.

□

For completeness we now briefly mention the momentum map of Ψ_S , which was found (in different disguise) by Kummer in [4]. Here \Im applied to a quaternion $a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ produces the 3-dimensional vector $(a, b, c)^t$.

Lemma 3.2. *The group action Ψ_S has momenta $\mathbf{L} = \Im(\frac{1}{2}\mathbf{Q} * \bar{\mathbf{P}} * \mathbf{k})$ and $L_\tau = \mathbf{Q}^T K \mathbf{P}$ which are mapped into the Lie algebra \mathfrak{g} of G by $\frac{1}{2}(\text{Isoc}(\hat{\mathbf{L}}) + K L_\tau)$. If in addition the bilinear relation is imposed, then $\pi \circ \mathbf{L}$ becomes the ordinary momentum $\mathbf{q} \times \mathbf{p}$.*

Proof. The ODE whose flow is Ψ_S with $S = \exp(Bt)$ is $\dot{\mathbf{Q}} = B\mathbf{Q}$, $\dot{\mathbf{P}} = B\mathbf{P}$, which comes from the Hamiltonian $\mathbf{Q}^T B \mathbf{P}$. Three components of the angular momentum are thus $L_\alpha =$

$\frac{1}{2}\mathbf{Q}^T \text{Isoc}(\hat{\mathbf{e}}_\alpha)\mathbf{P}$ where $\alpha \in \{x, y, z\}$ and \mathbf{e}_α is the unit vector in the direction of α . These components indeed form the first three components of $\frac{1}{2}\mathbf{Q} * \bar{\mathbf{P}} * \mathbf{k}$, for example, $\mathbf{Q}^T \text{Isoc}(\hat{\mathbf{e}}_y)\mathbf{P} = \frac{1}{2}(Q_1P_3 - P_1Q_3 + Q_2P_4 - P_2Q_4)$ is the \mathbf{i} -component of the quaternion $\frac{1}{2}\mathbf{Q} * \bar{\mathbf{P}} * \mathbf{k}$.

The second statement is shown through direct computation. See [4] for more details.

□

In this section, the final simple but crucial observation is that Ψ_S has four simple quadratic invariant polynomials, which will form the new coordinates in the singular reduction.

Lemma 3.3. *The basic polynomial invariants of the group action Ψ_S of G are*

$$X_1 = \mathbf{Q}^T \mathbf{Q} / \sqrt{2}, \quad X_2 = \mathbf{P}^T \mathbf{P} / \sqrt{2}, \quad X_3 = \mathbf{Q}^T \mathbf{P}, \quad X_4 = \mathbf{P}^T \mathbf{K} \mathbf{Q}.$$

The Poisson bracket of these invariants is closed and is the Lie-Poisson structure of $\mathfrak{u}(1, 1)$.

Proof. Firstly, $SO(4)$, as the group of rotations preserves the inner product on \mathbb{R}^4 . Thus, G as subgroup of $SO(4)$ must also preserve the inner product. Hence, X_1, X_2, X_3 are clearly invariants.

Since ψ_S acts in the same way on \mathbf{Q} and \mathbf{P} , in order to find invariant quadratic forms it is enough to consider invariant forms $\mathbf{U}^T \mathbf{M} \mathbf{V} = (\mathbf{S} \mathbf{U})^T \mathbf{S} \mathbf{M} \mathbf{V} = \mathbf{U}^T \mathbf{S}^t \mathbf{M} \mathbf{S} \mathbf{V}$ for arbitrary vectors \mathbf{U}, \mathbf{V} . If $\mathbf{M} = \mathbf{S}^t \mathbf{M} \mathbf{S}$ holds for some \mathbf{M} then there are three invariant quadratic forms given by $\mathbf{Q}^T \mathbf{M} \mathbf{Q}$, $\mathbf{Q}^T \mathbf{M} \mathbf{P}$, and $\mathbf{P}^T \mathbf{M} \mathbf{P}$.

Now $\mathbf{S} = \exp(\mathbf{B}t)$, and differentiating at 0 implies that $\mathbf{B} \mathbf{M} = \mathbf{M} \mathbf{B}$. This has only two independent solutions, $\mathbf{M} = \mathbf{I}$ and $\mathbf{M} = \mathbf{K}$. The antisymmetric $\mathbf{M} = \mathbf{K}$ only produces one non-zero invariant $X_4 = \mathbf{Q}^T \mathbf{K} \mathbf{P}$, and $\mathbf{M} = \mathbf{I}$ reproduces the three scalar product invariants already mentioned. Equivalently, one can show by direct computation that the only quadratic forms that simultaneously have vanishing Poisson bracket with L_α for all $\alpha \in \{x, y, z, \tau\}$ are in the span of X_1, \dots, X_4 . Therefore, the set X_1, \dots, X_4 is a basis for the vector space of quadratic invariants. The only non-vanishing Poisson brackets are

$$\{X_1, X_2\} = 2X_3, \quad \{X_2, X_3\} = -2X_2, \quad \{X_3, X_1\} = -2X_1.$$

Clearly the invariants X_1, \dots, X_4 are closed under the Poisson bracket.

Using (X_1, X_2, X_3, X_4) as a basis for the space of quadratic invariants, the Poisson structure

matrix is

$$\begin{pmatrix} 0 & 2X_3 & 2X_1 & 0 \\ -2X_3 & 0 & -2X_2 & 0 \\ -2X_1 & 2X_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with Casimir X_4 , the bi-linear relation.

The algebra $\mathfrak{u}(1, 1; iJ_2)$ is the set of complex matrices M that satisfy $(H_1 M)^\dagger + H_1 M = 0$ for the hermitian matrix $H_1 = iJ_2$, where J_2 is the standard symplectic 2×2 matrix, so that H_1 has eigenvalues $1, -1$, and hence signature $(1, 1)$. If we chose

$$b_1 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b_4 = -iI$$

as a basis for $\mathfrak{u}(1, 1)$, then the algebra of commutators of b_i is identical to the algebra of Poisson brackets of the X_i .

□

A relation of the regularised Kepler problem to $\mathfrak{u}(2, 2)$ can be found in [4], but this is not directly related to our $\mathfrak{u}(1, 1)$, which is the algebra of the quadratic invariants. Note that X_4 is not only an invariant but also a generator. The other generators are the component of L , and are not invariant under Ψ_S . However, the sum of their squares is, and hence can be written in terms of the above invariants: $L_x^2 + L_y^2 + L_z^2 = \frac{1}{2}X_1X_2 - \frac{1}{4}X_3^2$.

The normalisation of the basis and the invariants is chosen so that basis vectors are normalised with respect to the scalar product $\langle A, B \rangle = \text{Tr}(A^\dagger B)/2$. This ensures that the Lax form of the equations $\dot{L} = [P, L]$, which we are now going to derive, is particularly symmetric. The Hamiltonian in terms of the quadratic invariants is a linear function

$$H = \frac{1}{8\mu}X_2 - m_1m_2 - hX_1.$$

and the equations of motion are linear as well and given by

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{pmatrix} = \begin{pmatrix} 0 & 2X_3 & 2X_1 \\ -2X_3 & 0 & -2X_2 \\ -2X_1 & 2X_2 & 0 \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}, \quad (3.1)$$

where $H_i = \partial H / \partial X_i$. Since X_4 commutes with all X_i we can ignore it. On the level of the algebra we reduce by the centre, and get $\mathfrak{su}(1, 1)$. To emphasise the $\mathfrak{su}(1, 1; iJ_1) = \mathfrak{sl}(2, \mathbb{R})$

structure these equations can be written in Lax form by defining

$$L = J_2 \begin{pmatrix} \sqrt{2}X_1 & X_3 \\ X_3 & \sqrt{2}X_2 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \sqrt{2}H_1 & H_3 \\ H_3 & \sqrt{2}H_2 \end{pmatrix} J_2 \quad (3.2)$$

where J_2 is the standard symplectic 2×2 matrix, so that the equations of motion (3.1) are equivalent to

$$\dot{L} = [P, L].$$

This is yet another way to write the regularised equations of the Kepler problem. We recover the angular momentum as the Casimir $\det L = 2X_1X_2 - X_3^2$. In the case of 3 or more bodies the Lax form of the equation gives non-trivial additional information on the Casimirs, see below.

Notice that the symmetry reduction using invariants of the un-regularised Kepler problem leads to a Poisson structure of $\mathfrak{sl}(2, \mathbb{R})$ as well, see, e.g. [1], however, with a different (non-regularised) Hamiltonian.

4 The 3-body problem, $n = 3$

The G -action Ψ_S on pairs (\mathbf{q}, \mathbf{p}) extends to an action (denoted by the same letter) on triples of pairs $(\mathbf{q}_{ij}, \mathbf{p}_{ij})$. Since the action acts diagonally, to get the corresponding angular momenta the individual momenta are simply added together, $\mathcal{L}_a = \sum L_a^i$ for $a \in \{x, y, z\}$. In this way the action projects down by π to the usual action of the angular momentum.

Choosing the correct symmetry group is crucial in order to obtain a good set of quadratic invariants. Since any flow generated by L_τ^i is annihilated by π there is a choice in defining the symmetry group and its action. We could define an action of $SU(2) \times SO(2) \times SO(2) \times SO(2)$ where the action of each $SO(2)$ is the flow generated by L_τ^i , $i = 1, 2, 3$ for the three particles. The set of quadratic invariants is then much smaller, since the group is larger. However, Heggie's Hamiltonian cannot be written in terms of these 9 quadratic invariants, even though it is clearly invariant under it. Instead of working with higher degree invariants, we prefer to stick to quadratic invariants and instead consider a smaller group action. Hence we keep the same group $G = SU(2) \times SO(2)$ and let $SO(2)$ act diagonally on the three particles. The corresponding flow is generated by $\mathcal{L}_\tau = \sum L_\tau^i$. With this choice of extended G -action Ψ_S gives the smallest set of closed quadratic invariants in terms of which the Hamiltonian can be expressed.

Lemma 4.1. *A quadratic form $Q = \mathbf{X}^T M \mathbf{X}$ that is invariant under Ψ_S has matrix*

$$M = [W]_{\text{sym}} \otimes I_4 + [W]_{\text{skew}} \otimes K$$

where W is an arbitrary 6×6 matrix, $\mathbf{X} = (\mathbf{Q}_1^T, \mathbf{Q}_2^T, \mathbf{Q}_3^T, \mathbf{P}_1^T, \mathbf{P}_2^T, \mathbf{P}_3^T)^T$ and \otimes denotes the Kronecker product. The vector space of quadratic invariants of this form is closed under the Poisson bracket.

Proof. Since Ψ_S acts diagonally, the arguments from Lemma 3.3 can be repeated. Hence the invariant quadratic forms are either of the form $\alpha_{ij} = \mathbf{Q}_i^T \mathbf{Q}_j g_{ij}$, $\beta_{ij} = \mathbf{P}_i^T \mathbf{P}_j g_{ij}$, $\gamma_{ij} = \mathbf{Q}_i^T \mathbf{P}_j$, where $g_{ij} = 1/\sqrt{2}$ for $i = j$ and 1 otherwise, or they involve the matrix K and are $a_{ij} = \mathbf{Q}_i^T K \mathbf{Q}_j$, $b_{ij} = \mathbf{P}_i^T K \mathbf{P}_j$, $c_{ij} = \mathbf{Q}_i^T K \mathbf{P}_j$. The first group has 21 elements, and the second group has 15 elements because the expressions are identically zero when $i = j$. Any quadratic form on phase space can be written as $\mathbf{X}^T M \mathbf{X}$. For the first group of 24 invariant quadratic forms we have $M = S \otimes I_4$ where $S \in \text{sym}(6)$. Similarly, for the second group of quadratic forms over K we have $M = A \otimes K$ where $A \in \text{skew}(6)$. As the sum of invariants is invariant, the matrix for any quadratic invariant can be written as $S \otimes I_4 + A \otimes K$. Thus, the set of quadratic invariants is of the form $Q = \mathbf{X}^T M \mathbf{X}$ where

$$2M = [W]_{\text{sym}} \otimes I_4 + [W]_{\text{skew}} \otimes K \quad (4.1)$$

where W is an arbitrary 6×6 matrix and so the space of quadratic invariants is isomorphic to $\text{Mat}(6 \times 6, \mathbb{R})$ as a vector space. Let $\mathbf{X}^T M \mathbf{X}$, and $\mathbf{X}^T N \mathbf{X}$ be two arbitrary quadratic forms. Then the Poisson bracket induces an algebra on symmetric matrices given by

$$M * N = 2[MJN]_{\text{sym}} = MJN - NJM. \quad (4.2)$$

It is well known that for general symmetric matrices this algebra is $\mathfrak{sp}(m)$ where $m = \dim(\mathbf{X})$. In our case we have a sub-algebra of matrices of the form (4.1), say $2M = \tilde{A} \otimes I_4 + \check{A} \otimes K$, $2N = \tilde{B} \otimes I_4 + \check{B} \otimes K$, and using $J = J_6 \otimes I_4$, where $\tilde{(\cdot)} = [(\cdot)]_{\text{sym}}$ and $\check{(\cdot)} = [(\cdot)]_{\text{skew}}$ we find

$$2M * N = [(\tilde{A} \otimes I_4 + \check{A} \otimes K)(J_6 \otimes I_4)(\tilde{B} \otimes I_4 + \check{B} \otimes K)]_{\text{sym}} \quad (4.3)$$

$$= [(\tilde{A} J_6 \tilde{B} - \check{A} J_6 \check{B}) \otimes I_4 + (\check{A} J_6 \tilde{B} + \tilde{A} J_6 \check{B}) \otimes K]_{\text{sym}} \quad (4.4)$$

$$= [\tilde{A} J_6 \tilde{B} - \check{A} J_6 \check{B}]_{\text{sym}} \otimes I_4 + [\check{A} J_6 \tilde{B} + \tilde{A} J_6 \check{B}]_{\text{skew}} \otimes K \quad (4.5)$$

so that this sub-algebra, and hence the Poisson bracket of quadratic invariants of the form (4.1), is closed. Note that the Kronecker product of two antisymmetric matrices is symmetric. As particular examples of the above general rule we have, e.g., that $\{\alpha_{1,1}, \beta_{1,1}\} = 2\gamma_{1,1}$, and $\{\alpha_{1,1}, c_{3,1}\} = -\sqrt{2}a_{1,3}$. By setting the antisymmetric parts \tilde{A} and \tilde{B} to zero, it is clear that the 21-dimensional subspace spanned by α, β, γ is closed under the Poisson bracket and hence forms a sub-algebra within the sub-algebra of invariant quadratic forms.

□

Defining $f_{ij} = 4(\gamma_{i,j}\gamma_{j,i} - \gamma_{i,i}\gamma_{j,j} + \beta_{i,j}\alpha_{i,j} - c_{i,j}c_{j,i} + b_{i,j}a_{i,j})$, the Hamiltonian in terms of the invariant quadratic forms reads

$$\begin{aligned} H = & \frac{1}{8} \left(\frac{\alpha_{2,2}\alpha_{3,3}}{\mu_{23}}\beta_{1,1} + \frac{\alpha_{3,3}\alpha_{1,1}}{\mu_{13}}\beta_{2,2} + \frac{\alpha_{1,1}\alpha_{2,2}}{\mu_{12}}\beta_{3,3} \right) \\ & - \frac{1}{16} \left(\frac{\alpha_{1,1}}{m_1}f_{23} + \frac{\alpha_{2,2}}{m_2}f_{13} + \frac{\alpha_{3,3}}{m_3}f_{12} \right) \\ & - m_2m_3\alpha_{2,2}\alpha_{3,3} - m_1m_3\alpha_{1,1}\alpha_{3,3} - m_1m_2\alpha_{1,1}\alpha_{2,2} - h\alpha_{1,1}\alpha_{2,2}\alpha_{3,3} . \end{aligned}$$

Using this Hamiltonian we can now write down the regularised symmetry reduced 3-body dynamics as $\dot{f} = \{f, H\}$, where f is any function of the 36 invariants. It is rather unfortunate that the dimension of the space of invariants is bigger than the dimension of the original phase space, so from the point of view of efficiency of numerical integration nothing can be gained here.

In order to work out the isomorphism type of the Lie algebra of quadratic invariants, we first induce a Lie bracket $[\cdot, \cdot]_m$ on $\text{Mat}(6 \times 6, \mathbb{R})$ using the Poisson bracket. We then show that this bracket is isomorphic to $\mathfrak{u}(3, 3)$ with the standard commutator bracket.

The Lie-algebra of quadratic invariants (respectively of their symmetric matrices) defined in (4.2) induces a Lie-algebra on $\text{Mat}(6 \times 6, \mathbb{R})$

simply by reading off the first factors of the Kronecker product in (4.5), thus we define

$$[\tilde{A} + \check{A}, \tilde{B} + \check{B}]_m = 2[\tilde{A}J_6\tilde{B} - \check{A}J_6\check{B}]_{\text{sym}} + 2[\check{A}J_6\tilde{B} + \tilde{A}J_6\check{B}]_{\text{skew}} . \quad (4.6)$$

This leads us to the main theorem of this paper:

Theorem 4.1. *The symmetry reduced regularised 3-body problem has a Lie-Poisson structure with algebra $\mathfrak{u}(3, 3)$ and a corresponding Hilbert basis of 36 quadratic functions invariant under Ψ_S .*

Proof. For $A \in \text{Mat}(6 \times 6, \mathbb{R})$ the matrix $M = J(\tilde{A} + i\check{A})$ is in $\mathfrak{u}(3, 3)$. Here the indefinite Hermitian algebra is defined with respect to the indefinite Hermitian matrix $H = iJ$ with eigenvalues ± 1 each with multiplicity 3 so that the signature is $(3, 3)$. Now it is easy to check that $(HM)^\dagger + HM = 0$, and that matrices of the form $J(\tilde{A} + i\check{A})$ are closed under the commutator, and hence are in $\mathfrak{u}(3, 3)$.

Now we show that the vector space isomorphism $h : \text{Mat}(6 \times 6, \mathbb{R}) \rightarrow \mathfrak{u}(3, 3)$ with $h(A) = J_6(\tilde{A} + i\check{A})$ is in fact an isomorphism of Lie algebras: $h([A, B]_m) = [h(A), h(B)]$. First notice that the bracket $[\cdot, \cdot]_m$ from (4.6) can be rewritten as

$$[A, B]_m = [\tilde{A} + \check{A}, \tilde{B} + \check{B}]_m = -J_6([J_6\tilde{A}, J_6\tilde{B}] - [J_6\check{A}, J_6\check{B}] + [J_6\tilde{A}, J_6\check{B}] + [J_6\check{A}, J_6\tilde{B}]) ,$$

so that $[[A, B]_m]_{sym} = -J_6([J_6\tilde{A}, J_6\tilde{B}] - [J_6\check{A}, J_6\check{B}])$ and $[[A, B]_m]_{skew} = -J_6([J_6\tilde{A}, J_6\check{B}] + [J_6\check{A}, J_6\tilde{B}])$. Hence, on the one hand we have

$$h([A, B]_m) = J_6(-J_6([J_6\tilde{A}, J_6\tilde{B}] - [J_6\check{A}, J_6\check{B}]) + i(-J_6([J_6\tilde{A}, J_6\check{B}] + [J_6\check{A}, J_6\tilde{B}]))).$$

On the other hand we have

$$[h(A), h(B)] = [J_6(\tilde{A} + i\check{A}), J_6(\tilde{B} + i\check{B})]$$

and expanding the commutator and collecting real and imaginary parts shows that in deed this equals $h([A, B]_m)$. This proves that space of quadratic invariants and $\mathfrak{u}(3, 3)$ are isomorphic as Lie algebras. □

In the final step we use the isomorphism just established to write the equations of motion in Lax form $\dot{L} = [P, L]$, using the Lie-Poisson bracket on $\mathfrak{u}(3, 3)$. This brings out most clearly the Casimirs of the reduced system, which are the traces of powers of L , or, alternatively, the coefficients of the characteristic polynomial of L . Note that the Lax form gives only 6 invariants, but since there are 36 variables the system is by no means integrable.

Lemma 4.2. *The Poisson structure has 6 Casimirs of degree 1 through 6. The linear Casimir is the sum of the bilinear integrals \mathcal{L}_τ , the quadratic Casimir is the sum of the three angular momenta squared $\mathcal{L}_x^2 + \mathcal{L}_y^2 + \mathcal{L}_z^2$.*

Proof. The Poisson bracket of the Lie algebra, in this matrix representation, can be written as

$$\{f, g\}(M) = \left\langle M, \left[\frac{df}{dM}, \frac{dg}{dM} \right] \right\rangle$$

where, the inner product is given by $\langle M, N \rangle = \text{Tr}(M^\dagger N)/2$ and $\frac{df}{dM}$ refers to the element in \mathfrak{g} that satisfies

$$\lim_{\varepsilon \rightarrow 0} [f(M + \varepsilon dM) - f(M)] = \left\langle dM, \frac{df}{dM} \right\rangle,$$

see, e.g., [9] for more details. The reason for choosing a normalised basis is that with respect to a normalised basis this can be written in the simple form $\dot{L} = [P, L]$, see below. Now the co-efficients of the characteristic polynomial of L are in fact the Casimirs of the Poisson bracket. The co-efficient of the fifth order term is just the sum of the bilinear integrals, \mathcal{L}_τ . The coefficient of the quartic term is

$$\mathcal{L}_x^2 + \mathcal{L}_y^2 + \mathcal{L}_z^2 + f(\mathcal{L}_\tau)$$

where $f(\mathcal{L}_\tau)$ is a quadratic function of the bilinear integrals. Under the reduction by the centre, this Casimir simply becomes $\mathcal{L}_x^2 + \mathcal{L}_y^2 + \mathcal{L}_z^2$.

□

Define

$$M = \begin{pmatrix} \sqrt{2}\alpha_{1,1} & \alpha_{1,2} + ia_{1,2} & \alpha_{1,3} + ia_{1,3} & \gamma_{1,1} + ic_{1,1} & \gamma_{1,2} + ic_{1,2} & \gamma_{1,3} + ic_{1,3} \\ \alpha_{1,2} - ia_{1,2} & \sqrt{2}\alpha_{2,2} & \alpha_{2,3} + ia_{2,3} & \gamma_{2,1} + ic_{2,1} & \gamma_{2,2} + ic_{2,2} & \gamma_{2,3} + ic_{2,3} \\ \alpha_{1,3} - ia_{1,3} & \alpha_{2,3} - ia_{2,3} & \sqrt{2}\alpha_{3,3} & \gamma_{3,1} + ic_{3,1} & \gamma_{3,2} + ic_{3,2} & \gamma_{3,3} + ic_{3,3} \\ \gamma_{1,1} - ic_{1,1} & \gamma_{2,1} - ic_{2,1} & \gamma_{3,1} - ic_{3,1} & \sqrt{2}\beta_{1,1} & \beta_{1,2} + ib_{1,2} & \beta_{1,3} + ib_{1,3} \\ \gamma_{1,2} - ic_{1,2} & \gamma_{2,2} - ic_{2,2} & \gamma_{3,2} - ic_{3,2} & \beta_{1,2} - ib_{1,2} & \sqrt{2}\beta_{2,2} & \beta_{2,3} + ib_{2,3} \\ \gamma_{1,3} - ic_{1,3} & \gamma_{2,3} - ic_{2,3} & \gamma_{3,3} - ic_{3,3} & \beta_{1,3} - ib_{1,3} & \beta_{2,3} - ib_{2,3} & \sqrt{2}\beta_{3,3} \end{pmatrix}$$

and

$$L = J_6 M, \quad P = dM J_6,$$

so that the equations of motion of the symmetry reduced regularised spatial 3-body problem can be written in Lax form

$$\dot{L} = [P, L].$$

Because we have chosen a self-dual basis the matrix dM is simply given by replacing each entry in M by the derivative of the Hamiltonian H with respect to the variables of that entry, compare (3.2). Reduction by the centre of the algebra which is generated by the Linear Casimir \mathcal{L}_τ gives $\mathfrak{su}(3, 3)$. This can be achieved by subtracting $\text{Tr } L = -2i \sum c_{i,i}$ in the diagonal of L , but the equations are more symmetric if we stay in $\mathfrak{u}(3, 3)$.

The fact that the three difference vectors \mathbf{q}_{ij} add to zero induces three additional quadratic integrals T_1, T_2, T_3 . The flow of these integrals is non-compact, and we were not able to use it for symmetry reduction. The three momenta $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z$, and the integrals T_i form the Algebra $\mathfrak{se}(3)$.

5 The n -body problem

Theorem 5.1. *The symmetry reduced regularised n -body problem has a Lie-Poisson structure with algebra $\mathfrak{u}(m, m)$ where $m = n(n-1)/2$.*

Proof. As shown in Lemma 4.1, the nature of the invariants under Ψ_S are independent of the number of particles. They are realised as in the aforementioned lemma in phase space by the use of symmetric and antisymmetric matrices of size $2m \times 2m$ where m denotes the number

of difference vectors in the system. This establishes the vector space isomorphism to the space of $2m \times 2m$ matrices. Furthermore, by Theorem 4.1, it is apparent that the Lie algebra of invariants is isomorphic to $\mathfrak{u}(m, m)$. As m is equal to $\binom{n}{2} = n(n-1)/2$, the algebra of invariants for the symmetry reduced regularised n -body problem has a Lie-Poisson structure with algebra $\mathfrak{u}(n(n-1)/2, n(n-1)/2)$.

□

6 Conclusion

In this paper, we have shown that the quadratic invariants of the regularised n -body problem are either inner products or quadratic forms over the antisymmetric matrix K . These invariants form a Lie-Poisson algebra that is isomorphic to the Lie algebra $\mathfrak{u}(m, m)$ where $m = n(n-1)/2$ which is the algebra corresponding to the group that preserves hermitian forms of signature (m, m) . The dimension of this Lie Algebra is of order n^4 . Thus the use of such an algebra to obtain numerical solutions is improbable for large values of n . Despite this, the isomorphism to $\mathfrak{u}(m, m)$ yields a large amount of information about the rich structure of these invariants and provides insight into the n -body problem.

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